SEPARATING LAYER IN HIGH-TEMPERATURE STREAMS

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UDC 532.526

An exact solution was constructed in [1] for a high-temperature axisymmetric jet flow for Prandtl numbers less than or equal to one. It is shown in this paper that this exact solution is acceptable in a Prandtl number range from 1 to 3 but in a bounded domain of variation of the radial coordinate.

1. The problem describing the efflux of a high-temperature jet from a cylindrical hole has the following form within the framework of boundary-layer theory:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)=\rho\left(v\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial \zeta}\right), \tag{1.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v)+\frac{\partial}{\partial \zeta}(\rho w)=0,\ \rho T=1,\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\Pr\rho\left(v\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial \zeta}\right);$$

$$\partial w/\partial r=v=\partial T/\partial r=0 \text{ for } r=0; \tag{1.2}$$

$$w=0,\ T=\varepsilon \text{ for } r\to\infty, \tag{1.3}$$

where r, ζ Re are cylindrical coordinates (r and ζ are the internal coordinates in the asymptotic expansion in the small parameter $\mathrm{Re}^{-1}$), $\mathrm{Re}=\sqrt{\rho_m I_{1m}/2\pi}/\mu_m$ is a certain analog of the Reynolds number; w and vRe are the axial and radial velocity components; T is the temperature, ρ is the density, $\Pr = c_{pm}\mu_m/\lambda_m$ is the Prandtl number. The temperature $T_m$, density $\rho_m$, specific heat at constant pressure $c_{pm}$, heat conduction $\lambda_m$, dynamic viscosity $\mu_m$, total momentum flux $I_{1m}$, and total enthalpy flux $I_{2m}$ scales are considered given:

$$I_{1m}=2\pi\rho_m V_m^2 L_m^2\int_0^\infty \rho w^2 r dr,\ I_{2m}=2\pi c_{pm}\rho_m T_m V_m L_m^2\int_0^\infty \rho(T-\varepsilon)wr dr.$$

Selected as velocity and length scales, respectively, are

$$V_m=c_{pm}T_m I_{1m}/I_{2m},\ L_m=I_{2m}/(c_{pm}T_m\sqrt{2\pi\rho_m I_{1m}}).$$

It was assumed in the system (1.1) that the dynamic viscosity, heat conductivity, and specific heat are constants. Not formulated in problem (1.1)-(1.3) are the initial conditions at $\zeta=\zeta_0$; this is because self-similar solutions will be examined that are suitable at large distances from the mouth of the jet (or solutions for point sources). In this case, problem (1.1)-(1.3) must be closed by integral conditions for the conservation of the momentum and the enthalpy flux

$$\int_0^\infty \rho w^2 r dr=1,\ \int_0^\infty \rho w(T-\varepsilon) r dr. \tag{1.4}$$

It is expedient to go from the variables r, ζ to the variables x, ζ in the problem (1.1)-(1.4), where

$$x=r/\sqrt{\zeta}, \tag{1.5}$$

and to introduce s instead of the function v by means of the formula

$$s=\sqrt{\zeta}v-(1/2)xw. \tag{1.6}$$

By using (1.5) and (1.6) the problem (1.1)-(1.4) is converted to the form

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right)=\rho\left(s\frac{\partial w}{\partial x}+\zeta w\frac{\partial w}{\partial \zeta}\right), \tag{1.7}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 138-146, November-December, 1983. Original article submitted October 26, 1982.

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$$\frac{1}{x}\frac{\partial}{\partial x}(x\rho s)+\frac{\partial}{\partial\zeta}(\zeta\rho w)=0,\quad \rho T=1,$$
$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right)=\Pr\rho\left(s\frac{\partial T}{\partial x}+\zeta w\frac{\partial T}{\partial\zeta}\right); \tag{1.8}$$
$$\partial w/\partial x=s=\partial T/\partial x=0 \text{ for } x=0;$$
$$w=0,\ T=\varepsilon \text{ for } x\to\infty; \tag{1.9}$$
$$\int_0^\infty \zeta\rho w^2 x dx=1,\ \int_0^\infty \zeta\rho w(T-\varepsilon)x dx=1. \tag{1.10}$$

We will consider problem (1.7)-(1.10) in the case when $\varepsilon \ll 1$, i.e., we shall study the flow domain where the temperature on the jet axis is considerably higher than the temperature at infinity. The ideas and terminology of the theory of sewn-together asymptotic expansions (see [2], say) will be used in constructing the asymptotic expansions, formulating the limits, and setting the boundary conditions

$$\varepsilon\to 0,\ x,\ \zeta \text{ are fixed} \tag{1.11}$$

Then the following asymptotic expansions of the solutions of problem (1.7)-(1.10) which are suitable near the boundary r = 0 can be constructed

$$w(x,\zeta;\varepsilon)=w_0(x,\zeta)+\nu_{w1}(\varepsilon)w_1(x,\zeta)+\nu_{w2}(\varepsilon)w_2(x,\zeta)+\ldots, \tag{1.12}$$
$$T(x,\zeta;\varepsilon)=T_0(x,\zeta)+\nu_{T1}(\varepsilon)T_1(x,\zeta)+\ldots,\ \rho(x,\zeta;\varepsilon)=\rho_0(x,\zeta)+\ldots,$$
$$s(x,\zeta;\varepsilon)=s_0(x,\zeta)+\nu_{s1}(\varepsilon)s_1(x,\zeta)+\ldots,$$

here

$$\frac{\nu_{w,n+1}(\varepsilon)}{\nu_{wn}(\varepsilon)}\to 0,\ \frac{\nu_{T,n+1}(\varepsilon)}{\nu_{Tn}(\varepsilon)}\to 0,\ldots \text{ for } \varepsilon\to 0\ (n=0,1,2,\ldots). \tag{1.13}$$

We call the problem in the zeroth approximation in $\varepsilon$ when expansion (1.12) is substituted into (1.7)-(1.10) the problem of the hot boundary layer (the temperature in this domain is considerably higher than the temperature at infinity). The system of equations (1.7) and the boundary conditions (1.8) remain invariant in this problem, and its nontrivial solution is [1]:

$$w_0=\frac{3-\Pr}{4\zeta}\left(1-\frac{x^2}{x_0^2}\right)^{2/(\Pr-1)},\ T_0=\frac{\Pr+1}{4\zeta}\left(1-\frac{x^2}{x_0^2}\right)^{2\Pr/(\Pr-1)},$$
$$s_0=-\frac{3-\Pr}{8\zeta}\left(1-\frac{x^2}{x_0^2}\right)^{(\Pr+1)/(\Pr-1)}, \tag{1.14}$$

here $x_0=\sqrt{8(\Pr+1)/[(3-\Pr)(\Pr-1)]}$.] The solutions (1.14) are normalized according to the integral conditions (1.10) which have the following form for the zeroth approximation of the asymptotic expansion (1.12)

$$\int_0^{x_0}\zeta(w_0^2/T_0)x dx=1,\ \int_0^{x_0}\zeta w_0 x dx=1. \tag{1.15}$$

Let us note that the solutions (1.14) satisfy the integral conditions (1.15) only for Pr < 3. For Pr $\le$ 1 the solutions (1.14) are acceptable in the whole range of variation of the variable x $(0 \le x < \infty)$. It should hence be kept in mind that everything below refers to the Prandtl number range 1 < Pr < 3.

Solutions (1.14) do not satisfy the boundary conditions (1.9); moreover, the asymptotic expansion (1.12) becomes unacceptable near the surface $x = x_0$. In fact, in a zeroth approximation in $\varepsilon$ in (1.12) the temperature on this surface is zero, and according to boundary condition (1.9) the temperature equals $\varepsilon$ even at infinity, i.e., is greater than zero, or the zeroth approximation for the temperature in expansion (1.12) becomes less than the latter, which contradicts the principles of constructing the asymptotic expansions (1.13). Therefore, if it is assumed that the self-similar solution is valid in the neighborhood of the boundary r = 0, then near the surface $x = x_0$ another asymptotic expansion of problem (1.7)-

(1.10) must be constructed, where the zeroth term for the temperature in this expansion should equal $\varepsilon$ in order of magnitude.

2. Let us construct the asymptotic expansion of the solution of problem (1.7)-(1.10) which is suitable for the surface $x = x_0$, i.e., we assume that a thin (compared to the thickness of the hot boundary layer) layer exists near the surface $x = x_0$ (a domain of a sharp gradient of the functions). We later call this layer separating (it separates the high-temperature compressible gas flow domain with temperature $T \sim 1$ from the cold incompressible gas flow domain with temperature $T \sim \varepsilon$). Let us formulate the limit process for the separating layer:

$$y = (x - x_0)/\tilde{\delta}(\varepsilon), \ \zeta \text{ are fixed } \tilde{\delta}(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.1}$$

here $\tilde{\delta}(\varepsilon)$ can be interpreted as a quantity characterizing the thickness of the separating layer. The asymptotic expansion of the solutions of problem (1.7)-(1.10) can be represented in the form

$$\begin{aligned} w(r, \zeta; \varepsilon) &= \tilde{\nu}_w(\varepsilon)\tilde{w}(y, \zeta) + \tilde{\nu}_{w1}(\varepsilon)\tilde{w}_1(y, \zeta) + \ldots, \\ T(r, \zeta; \varepsilon) &= \varepsilon\tilde{T}(y, \zeta) + \tilde{\nu}_{T1}(\varepsilon)\tilde{T}_1(y, \zeta) + \ldots, \\ s(r, \zeta; \varepsilon) &= \tilde{\nu}_s\tilde{s}(y, \zeta) + \tilde{\nu}_{s1}(\varepsilon)\tilde{s}_1(y, \zeta) + \ldots, \\ \rho(r, \zeta; \varepsilon) &= \varepsilon^{-1}\tilde{\rho}(y, \zeta) + \tilde{\nu}_{\rho1}(\varepsilon)\tilde{\rho}_1(y, \zeta) + \ldots, \end{aligned} \tag{2.2}$$

where $\tilde{\nu}_{wn}(\varepsilon)$, $\tilde{\nu}_{Tn}(\varepsilon)$, $\tilde{\nu}_{sn}(\varepsilon)$, and $\tilde{\nu}_{\rho n}(\varepsilon)$ satisfy the relationships analogous to (1.13) and the zero subscript is omitted from the zeroth terms. We define the functions of the small parameters $\tilde{\nu}_w$, $\tilde{\nu}_s$, $\tilde{\delta}$ in expressions (2.2) and (2.1) from the condition of merging the asymptotic expansion for the separating layer with the expansion for the hot boundary layer (1.12). To do this we formulate the limits intermediate between the limit for the hot boundary layer (1.11) and the limit for the separating layer (2.1). Let

$$\varkappa \to 0, \ \varepsilon/\varkappa \to 0, \ \tilde{\delta}(\varepsilon)/\tilde{\delta}(\varkappa) \to 0, \ y_\varkappa = (x - x_0)/\tilde{\delta}(\varkappa), \ \zeta \text{ are fixed} \\ \text{for} \ \varepsilon \to 0, \tag{2.3}$$

here $y_\varkappa < 0$. Then merging (2.2) and (1.11) at the intermediate limit (2.3), we have

$$\begin{aligned} \frac{3-\Pr}{4\zeta}\left(\frac{2}{x_0}\right)^a [-y_\varkappa\tilde{\delta}(\varkappa)]^a + O\left[\tilde{\delta}^{a+1}(\varkappa)\right] &= \tilde{\nu}_w(\varepsilon)\,\tilde{w}\left(\frac{y_\varkappa\tilde{\delta}(\varkappa)}{\tilde{\delta}(\varepsilon)}, \zeta\right) \\ &\quad + O\left[\tilde{\nu}_{w1}(\varepsilon)\right], \\ \frac{\Pr+1}{4\zeta}\left(\frac{2}{x_0}\right)^{a+2} [-y_\varkappa\tilde{\delta}(\varkappa)]^{a+2} + O\left[\tilde{\delta}^{a+3}(\varkappa)\right] &= \varepsilon\tilde{T}\left(\frac{y_\varkappa\tilde{\delta}(\varkappa)}{\tilde{\delta}(\varepsilon)}, \zeta\right) \\ &\quad + O\left[\tilde{\nu}_{T1}(\varepsilon)\right], \\ -\frac{3-\Pr}{2}\frac{x_0}{\zeta}\left(\frac{2}{x_0}\right)^{a+1} [-y_\varkappa\tilde{\delta}(\varkappa)]^{a+1} + O\left[\tilde{\delta}^{a+2}(\varkappa)\right] &= \tilde{\nu}_s(\varepsilon)\tilde{s}\left(\frac{y_\varkappa\tilde{\delta}(\varkappa)}{\tilde{\delta}(\varepsilon)}, \zeta\right) + O\left[\tilde{\nu}_{s1}(\varepsilon)\right], \end{aligned} \tag{2.4}$$

where $a = 2/(\Pr - 1)$, from which can be obtained successively

$$\tilde{\delta} = \varepsilon^{(\Pr-1)/(2\Pr)}, \ \tilde{\nu}_w = \varepsilon^{1/\Pr}, \ \tilde{\nu}_s = \varepsilon^{(\Pr+1)/(2\Pr)}. \tag{2.5}$$

As follows from (1.6), we note that the quantities w and v, in the zeroth approximation in $\varepsilon$ in expansion (2.2) are of the same order of smallness $v \sim w \sim \varepsilon^{1/\Pr}$ while the function $s \sim \varepsilon^{(\Pr+1)/(2\Pr)}$, i.e., s decreases more rapidly than v and w as $\varepsilon \to 0$. Hence, if problem (1.1)-(1.4) were considered initial in the construction of the asymptotic expansions, then to realize the successful merger procedure it would be insufficient to use just the zeroth terms of the expansions. Conversion of (1.6) and (1.5) would eliminate this difficulty.

Substituting series (2.2) into problem (1.7)-(1.9), and keeping in mind (2.1) and (2.5), we obtain a system of equations for the separating layer in the zeroth approximation in $\varepsilon$

$$\begin{aligned} &\frac{\partial^2\tilde{w}}{\partial y^2} = \tilde{\rho}\left(\tilde{s}\frac{\partial\tilde{w}}{\partial y} + \zeta\tilde{w}\frac{\partial\tilde{w}}{\partial\zeta}\right), \ \frac{\partial}{\partial y}(\tilde{\rho}\tilde{s}) + \frac{\partial}{\partial\zeta}(\zeta\tilde{\rho}\tilde{w}) = 0, \\ &\frac{\partial^2\tilde{T}}{\partial y^2} = \Pr\tilde{\rho}\left(\tilde{s}\frac{\partial\tilde{T}}{\partial y} + \zeta\tilde{w}\frac{\partial\tilde{T}}{\partial\zeta}\right), \ \tilde{\rho}\tilde{T} = 1. \end{aligned} \tag{2.6}$$

The boundary conditions for system (2.6) are obtained from the conditions for merging (2.2) and (1.11) at the intermediate limit (2.3) [see (2.4)]:

$$\lim_{y\to-\infty}\left[(-y)^{-a}\widetilde{w}(y,\zeta)\right]=\frac{3-\Pr}{4\zeta}\left(\frac{2}{x_0}\right)^a, \tag{2.7}$$

$$\lim_{y\to-\infty}\left[(-y)^{-a-2}\widetilde{T}(y,\zeta)\right]=\frac{1+\Pr}{4\zeta}\left(\frac{2}{x_0}\right)^{a+2},\quad \lim_{y\to-\infty}\left[(-y)^{-a-1}\widetilde{s}(y,\zeta)\right]=-\frac{3-\Pr}{\zeta}\left(\frac{2}{x_0}\right)^a,$$

while from conditions (1.9) as $y\to+\infty$,

$$\widetilde{w}=0,\ \widetilde{T}=1\ \text{for}\ y\to+\infty. \tag{2.8}$$

The problem (2.6)-(2.8) allows for a self-similar solution. Let us introduce new functions and new variables according to the formulas

$$\widetilde{w}(y,\zeta)=\zeta^{\widetilde{\alpha}_w}\widetilde{u}(\eta),\ \widetilde{T}(y,\zeta)=\zeta^{\widetilde{\alpha}_T}\widetilde{\theta}(\eta),\ \widetilde{s}(y,\zeta)=\zeta^{\widetilde{\alpha}_s}\widetilde{f}(\eta),\ y=\eta\zeta^{\widetilde{\alpha}}. \tag{2.9}$$

It follows from the boundary condition (2.8) for the temperature that

$$\widetilde{\alpha}_T=0, \tag{2.10}$$

while three equations

$$-1=-(a+2)\widetilde{\alpha},\ -1=\widetilde{\alpha}_w-a\widetilde{\alpha},\ -1=\widetilde{\alpha}_s-(a+1)\widetilde{\alpha}. \tag{2.11}$$

can be obtained from the boundary conditions (2.7). Let us note that the self-similarity conditions $\tilde{\alpha}_w=-2\tilde{\alpha}$, $\tilde{\alpha}_s=\tilde{\alpha}+\tilde{\alpha}_w$ can be obtained from the system (2.6); however, these conditions are a corollary of system (2.11). It is possible to find from (2.11)

$$\widetilde{\alpha}=(\Pr-1)/(2\Pr),\ \widetilde{\alpha}_w=-(\Pr-1)/\Pr,\ \widetilde{\alpha}_s=-(\Pr-1)/(2\Pr). \tag{2.12}$$

Substituting (2.9) into problem (2.6)-(2.8) and keeping (2.10)-(2.12) in mind, we obtain the system of equations

$$\widetilde{u}''=\widetilde{g}\widetilde{u}'+\widetilde{\alpha}_w\widetilde{u}^2/\widetilde{\theta},\ \widetilde{g}'=-\gamma\widetilde{u}/\widetilde{\theta},\ \widetilde{\theta}''=\Pr\widetilde{g}\widetilde{\theta}' \tag{2.13}$$

with the boundary conditions

$$\lim_{\eta\to-\infty}\left[(-\eta)^{-a}\widetilde{u}\right]=\frac{3-\Pr}{4}\left(\frac{2}{x_0}\right)^a,\ \lim_{\eta\to-\infty}\left[(-y)^{-a-2}\widetilde{\theta}\right]=\frac{1+\Pr}{4}\left(\frac{2}{x_0}\right)^{a+2},$$
$$\lim_{\eta\to-\infty}\left[(-\eta)\widetilde{g}\right]=-(\Pr+1)/[\Pr(\Pr-1)];\ \widetilde{u}=0,\ \widetilde{\theta}=1\ \text{for}\ \eta\to+\infty, \tag{2.14}$$

where the prime denotes the derivative with respect to $\tilde{\eta}$, $\tilde{g}(\eta)=\tilde{f}-\tilde{\alpha}\eta\tilde{u}$, $\tilde{\gamma}=(\Pr+1)/(2\Pr)$. System (2.13) allows a power-law solution of the form

$$\widetilde{u}=A(B\eta)^{\widetilde{a}},\ \widetilde{\theta}=\frac{A\Pr\widetilde{\gamma}}{\widetilde{a}+1}(B\eta)^{\widetilde{a}+2},\ \widetilde{g}=\frac{\widetilde{a}+1}{\Pr\eta}, \tag{2.15}$$

where A and B are arbitrary constants and the quantity $\tilde{a}$ is one of the roots of the cubic equation

$$(\Pr-1)\widetilde{\gamma}\widetilde{a}^3+\left[(\Pr-3)\widetilde{\gamma}-\widetilde{\alpha}_w\right]\widetilde{a}^2-\left[2\widetilde{\gamma}(\Pr+1)-3\widetilde{\alpha}_w\right]\widetilde{a}-2\widetilde{\alpha}_w=0. \tag{2.16}$$

It can be shown that for $\tilde{a}=a=2/(\Pr-1)$ Eq. (2.16) becomes an identity, i.e., (2.16) in the unknowns $\tilde{\alpha}$ and $\tilde{\alpha}_w$ becomes linearly dependent on the first two equations in system (2.11). Therefore, by selecting the appropriate constants A and B the solutions (2.15) for $\tilde{a}=a$ can be interpreted as asymptotic solutions of problem (2.13), (2.14) as $\eta\to-\infty$.

The solutions of problem (2.13), (2.14) as $\eta\to+$ tend to the asymptotic

$$\widetilde{u}=C_1\exp(\widetilde{g}_\infty\eta),\ \widetilde{\theta}=1+C_2\exp(\widetilde{g}_\infty\eta),\ g=\widetilde{g}_\infty=\text{const}<0, \tag{2.17}$$

where $C_1$ and $C_2$ are certain constants.

Problem (2.13), (2.14) was solved numerically for different Prandtl numbers. The curves $\tilde{u}(\eta)$, $\tilde{\theta}(\eta)$, $\tilde{g}(\eta)$ are displayed in Fig. 1 for Pr = 2. For the numerical integration the bound-

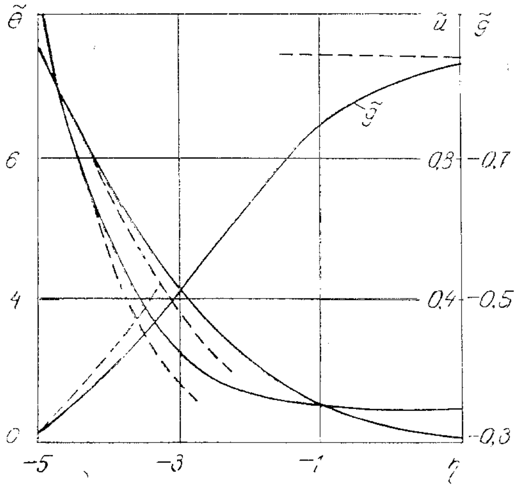


Fig. 1

ary conditions were posed at a certain large but finite distance. By varying this distance, the accuracy of the computation could be assessed when the singular boundary conditions are replaced by boundary conditions at a finite interval. The verifying computations permit the hope that the computational accuracy of the curves superposed in Fig. 1 is not lower than the accuracy of their graphical construction. We note that problem (2.13), (2.14) is invariant to the substitution of the variable $\eta \to \eta - \eta_0$, where $\eta_0$ is an arbitrary constant [an analogous remark can also be made for the problem (2.6)-(2.8)]. The quantity $\eta_0$ can be determined from the conditions for merging the succeeding terms of expansions (1.12) and (2.2). For the curves in Fig. 1, the quantity $\eta_0$ is selected from the condition $\tilde{u}(0) = 0.05$. It is seen from Fig. 1 that as $\eta \to \pm\infty$ the curves tend quite satisfactorily to the asymptotics described by (2.15) and (2.17) (the asymptotes are superposed by dashed lines). Let us note that the magnitude of the radial velocity undergoes a curious change in the separating layer. Thus, on the inner boundary of the separating layer $(y \to -\infty)$, $v \sim w \sim \varepsilon^{1/\mathrm{Pr}}$ while on the outer $(y \to +\infty)$, $v \sim s \sim \varepsilon^{(\mathrm{Pr}+1)/2\mathrm{Pr}}$ [see (1.6), (2.2), (2.5), (2.17)], i.e., as the quantity $\varepsilon$ diminishes the radial velocity on the outer boundary of the separating layer decreases considerably more rapidly than on the inner boundary.

3. Since problem (1.1)-(1.4) is cylindrically symmetric, it can then be expected that the solutions for w and v will damp out as $r \to \infty$ ($\zeta$ is fixed). However, it follows from the solution of the problem for the separating layer that the radial velocity outside this layer $(y \to +\infty)$ is independent of y [it must be recalled that $\tilde{w}$ decreases exponentially as $y \to +\infty$, and also we must return to the definition of the function $g(\eta)$]. In this connection, it is expedient to formulate the limit process for the domain $x > x_0$ in the form

$$\varepsilon \to 0,\ x,\ \zeta \text{ are fixed}, \quad \text{ьr}, \tag{3.1}$$

and to construct the appropriate asymptotic expansions of the solutions of the problem (1.7)-(1.10) in the small parameter $\varepsilon$, which will have the form

$$\begin{aligned} w(x, \zeta; \varepsilon) &= \bar{\nu}_w(\varepsilon)\bar{w}(x, \zeta) + \bar{\nu}_{w1}(\varepsilon)\bar{w}_1(x, \zeta) + \ldots, \\ T(x, \zeta; \varepsilon) &= \varepsilon + \bar{\nu}_{T1}(\varepsilon)\bar{T}_1(x, \zeta) + \ldots, \\ s(x, \zeta; \varepsilon) &= \varepsilon^{(\mathrm{Pr}+1)/2\mathrm{Pr}}\,\bar{s}(x, \zeta) + \bar{\nu}_{s1}(\varepsilon)\bar{s}_1(x, \zeta) + \ldots, \\ \rho(x, \zeta; \varepsilon) &= \varepsilon^{-1} + \bar{\nu}_{\rho 1}(\varepsilon)\bar{\rho}_1(x, \zeta) + \ldots, \end{aligned} \tag{3.2}$$

where $\bar{\nu}_w(\varepsilon) = O(\bar{\nu}_s(\varepsilon))$, while the zero subscript is omitted in the zeroth terms.

The following reasoning was used in constructing expansions (3.2), which we later call the expansions of the cold boundary layer, to determine the form of the zeroth terms $\bar{\nu}_w(\varepsilon)$, $\bar{\nu}_T(\varepsilon)$, $\bar{\nu}_s(\varepsilon)$. It follows from the expansions of problem (1.7)-(1.10) for the separating layer (2.2), formulas (2.5), and the behavior of these solutions as $y \to +\infty$ that the temperature and the function s become constants (independent of y), equal to $\varepsilon$ and $\varepsilon^{(\mathrm{Pr}+1)/2\mathrm{Pr}} \times \zeta^{-(\mathrm{Pr}-1)/2\mathrm{Pr}}\tilde{g}_\infty$, respectively, where $g_\infty$ is a certain constant known from the solution of problem (2.13), (2.14). Therefore, for successful realization of the merging procedure, it is

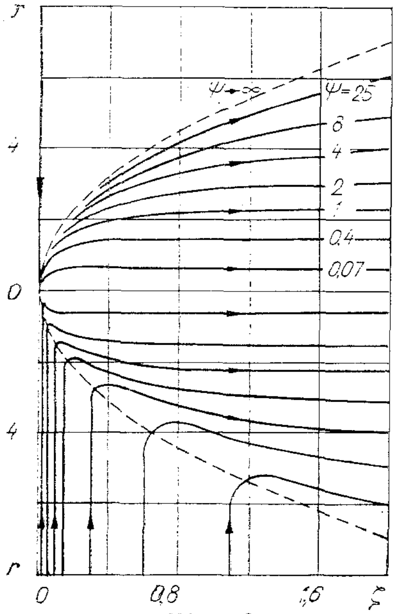

Fig. 2

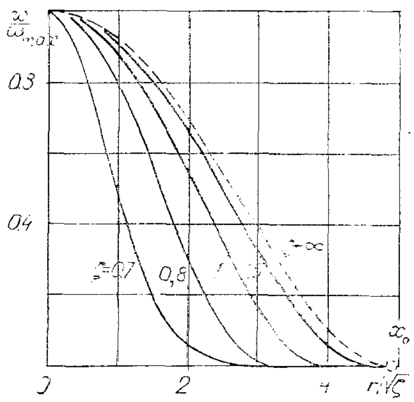

Fig. 3

necessary to set $\bar{\nu}_T(\varepsilon) = \varepsilon$, $\bar{\nu}_s(\varepsilon) = \varepsilon^{(Pr+1)/2Pr}$, where physical intuition insists that the temperature is a monotonic function of r or x (absence of heat sources and sinks). This means that $T(x, \zeta) = 1$ for the domain $x > x_0$, which does not contradict the energy conservation equation. The quantity $\bar{\nu}_w(\varepsilon)$ should be assumed small compared with $\bar{\nu}_s(\varepsilon)$ since, in the case of $\bar{\nu}_w = \bar{\nu}_s$ we obtain the following system of equations in the zeroth approximation in ε

$$\bar{s}\frac{\partial \bar{w}}{\partial x} + \zeta\bar{w}\frac{\partial \bar{w}}{\partial \zeta} = 0,\ \frac{1}{x}\frac{\partial}{\partial x}(x\bar{s}) + \frac{\partial}{\partial \zeta}(\zeta\bar{w}) = 0. \tag{3.3}$$

It can be shown that solutions satisfying the merger conditions and decreasing as $x \to \infty$ exist for the system (3.3). However, these solutions decrease insufficiently rapidly, and the integral $\int_{x_0}^{\infty} \bar{w}^2 x dx$ does not exist. Therefore, the asymptotic expansion for the axial velocity in the cold boundary layer should start with a term of a lesser order of smallness than $\varepsilon^{(Pr+1)/2Pr}$. The function $\bar{\nu}_w(\varepsilon)$ is apparently determined in the merging of the last terms of the expansions for the separating and cold boundary layers.

Substituting the asymptotic expansion (3.2) into system (1.7), and keeping the definition of the limit process in the form (3.1) in mind, we obtain an equation for $\bar{s}$ in the zeroth approximation in ε

$$\frac{1}{x}\frac{\partial}{\partial x}(x\bar{s}) = 0. \tag{3.4}$$

The boundary condition for (3.4) can be obtained in the merger of the expansions for the function s for the cold boundary layer and the expansions for the separating layer

$$\bar{s}|_{x=x_0} = \tilde{g}_\infty \zeta^{\tilde{\alpha}_s}, \tag{3.5}$$

here the quantity $\tilde{g}_\infty$ is considered known from the solution of problem (2.13) and (2.14). The solution of problem (3.4), (3.5) is evidently

$$\bar{s} = \tilde{g}_\infty \zeta^{\tilde{\alpha}_s} x_0/x. \tag{3.6}$$

In connection with the fact that $\bar{\nu}_w(\varepsilon) = o(\bar{\nu}_s(\varepsilon))$, the functions $\bar{s}$ for the cold boundary layer can be identified with $\sqrt{\zeta}v$ [see (1.6)], i.e., in the cold boundary layer the gas flows only along the radius, as follows from (3.6). Therefore, problem (1.7)-(1.11) is solved in the zeroth approximation in ε.

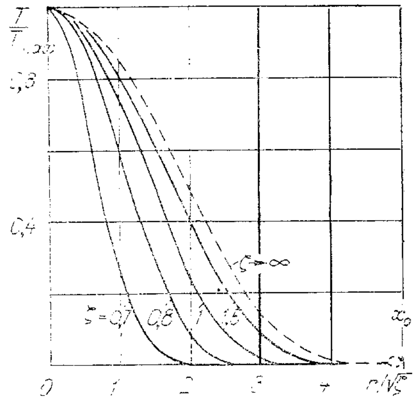

Fig. 4

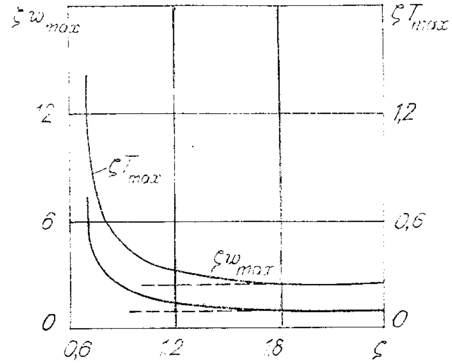

Fig. 5

To illustrate the flow in the high-temperature jet, we introduce the stream function

$$\psi = \int_0^r \rho w r dr. \tag{3.7}$$

Then the expression

$$r^2 = \frac{2(\mathrm{Pr}+1)}{3-\mathrm{Pr}} \frac{4\psi\zeta}{4\zeta + (\mathrm{Pr}-1)\psi}. \tag{3.8}$$

can be obtained for the stream line in the domain $r < x_0\sqrt{\zeta}$. It follows from (3.8) that as $\zeta \to \infty$ the stream lines become parallel to the axis $\zeta$ ($r^2 = 2(\mathrm{Pr} + 1)\psi/(3 - \mathrm{Pr})$), while as $\psi \to \infty$ the flow is along the interfacial surface $r = x_0\sqrt{\zeta}$, i.e., there is no gas flow outside this surface.

The stream line pattern defined by (3.8) is displayed in the upper part of Fig. 2. Let us note that the stream lines will have the same shape only for $\varepsilon \to 0$. For small but finite quantities $\varepsilon$ the stream line pattern will be different. Formula (3.8) becomes unacceptable near the surface $r = x_0\sqrt{\zeta}$. To find the stream function in this case, values of the appropriate functions for the separating layer (near the surface $r = x_0\sqrt{\zeta}$) or the cold boundary layer (outside the surface $r = x_0\sqrt{\zeta}$) must be substituted in (3.7). The stream line pattern for a small but finite value is displayed schematically in the lower part of Fig. 2. In this case the stream lines at distances from the jet axis scarcely greater than the thickness of the hot boundary layer become parallel to the coordinate direction r, i.e., the cold gas from infinity is set into motion along the radius, its heating and the change in its direction of motion occur only in the domain of the separating layer. Let us note that as $\varepsilon$ diminishes the stream lines outside the surface $r = x_0\sqrt{\zeta}$ will flatten out along the plane $\zeta = 0$ and as $\varepsilon \to 0$ all the stream lines (for $r > x_0\sqrt{\zeta}$) will be confluent at one $\zeta = 0$, as is displayed on the upper part of Fig. 2.

In order to confirm the correctness of the solution obtained, a numerical experiment was performed. Problem (1.1)-(1.3) was supplemented by the condition

$$w = w^0(r),\ T = T^0(r) \text{ for } \zeta = \zeta_0 \tag{3.9}$$

and solved numerically for different values of $\varepsilon$, the Prandtl number, and different modifications of assigning the initial conditions (3.9). The relative axial velocity and temperature profiles ($w_{max}$, $T_{max}$ are values of the respective velocity and temperature on the jet axis) are displayed in Figs. 3 and 4 for different values of $\zeta$ (these values are marked by numbers at the appropriate curves), while values of the product of the longitudinal coordinate $\xi$ by the velocity (temperature) on the axis are superposed in Fig. 5 as a function of $\zeta$. It is seen from the behavior of the curves in Figs. 3-5 that the velocity and temperature profiles become self-similar very rapidly as $\zeta$ increases. The computation of these curves was performed for the initial conditions (3.9) in the form

$$w = 2\exp(-r^2),\ T = 10.6\exp(-1.935r^2) + 3.45\cdot 10^{-5} \text{ for } \zeta = 0.7$$

and for Pr = 2, $\varepsilon = 3.45 \cdot 10^{-5}$. In connection with the invariance of the solutions of the problem relative to the change of variable $\zeta \to \zeta - \zeta_0$ ($\zeta_0$ is an arbitrary constant), the quantity $\zeta_0$ ($\zeta_0 = 0.7$) was selected from the condition of minimum in the difference of solutions of the problem (1.1)-(1.3), (3.9), and the self-similar solution (1.14).

In conclusion, some considerations should be expressed about the terminology used in this paper. Strictly speaking, no boundary layer (neither hot nor cold) occurs for the problem (1.7)-(1.10) as $\varepsilon \to 0$. The domain of nonuniform suitability of the asymptotic expansions of the solutions of this problem as $\varepsilon \to 0$ is not localized near the boundary (the boundary layer), but within the integration interval $0 \le r < \infty$. Such a situation usually occurs in considering nonuniformities of the internal boundary layer type [3]. However, characteristic for such a nonuniformity is that the internal boundary layer is defined: 1) for singularly perturbed systems of ordinary differential equations, where the singular perturbations are ordinarily understood to be a rise in the order of the perturbed system) 2) for boundary-value problems for which the boundary conditions are given in a finite interval. For problems with partial derivatives (1.7)-(1.10), the parameter $\varepsilon$ different from zero induces no perturbations into the problem that are usually called singular. The source of the nonuniformity in expansion (1.12) of the solutions of problem (1.7)-(1.10) is the presence of the infinite boundary. It is impossible to give boundary conditions arbitrarily for singular boundary-value problems. Conditions on the infinitely remote boundary should satisfy system (1.7). In the case of setting boundary conditions of the form

$$w \to 0,\ T \to 0 \text{ for } x \to \infty \tag{3.10}$$

an indeterminacy of the type 0/0 occurs in system (1.7). For instance, for Pr < 1 the asymptotic solutions of problem (1.7)-(1.10) as $x \to \infty$ will be

$$w = \frac{3-\mathrm{Pr}}{4\zeta}\left[\frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(\mathrm{Pr}+1)}x^2\right]^{-2/(1-\mathrm{Pr})},\ T = \frac{\mathrm{Pr}+1}{4\zeta}\left[\frac{(3-\mathrm{Pr})(1-\mathrm{Pr})}{8(\mathrm{Pr}+1)}x^2\right]^{-2\mathrm{Pr}/(1-\mathrm{Pr})},$$

i.e., for Pr < 1 substitution of conditions (3.10) is possible in system (1.7), while conditions (3.10) will be unsuitable here for Pr > 1. It is also necessary to recall that the considered problem (1.7)-(1.10) is itself a zeroth approximation in the asymptotic expansion in the small parameter $\mathrm{Re}^{-1}$ and, hence, the domain of nonuniformity of the expansion (1.12) is located in the boundary layer. Hence, in our opinion, the nonuniformity domain detected refers to a new class, and is called the separating layer.

The factorization method with iterations was used in numerical computations for problem (2.13), (2.14), and the method of [4] for problem (1.1)-(1.3), (3.9). The accuracy of computing the curves is not below the accuracy of their graphical display.

The author is grateful to M. A. Gol'dshtik, V. V. Pukhnachev, and V. N. Shtern for taking part in discussions of results of the research.

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